One-loop corrected thermodynamics of the extremal and nonextremal spinning Banados-Teitelboim-Zanelli black hole

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We consider the one-loop corrected geometry and thermodynamics of a rotating BTZ black hole by way of a dimensionally reduced dilaton model. The analysis begins with a comprehensive study of the non-extremal solution after which two different methods are invoked to study the extremal case. The first approach considers the extremal limit of the non-extremal calculations, whereas the second treatment is based on the following conjecture: extremal and non-extremal black holes are qualitatively distinct entities. We show that only the latter method yields regularity and consistency at the one-loop level. This is suggestive of a generalized third law of thermodynamics that forbids continuous evolution from non-extremal to extremal black hole geometries.

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I. INTRODUCTION

Nearly 30 years have passed since Bekenstein and Hawking conjectured the laws of black hole mechanics to be in analogy with those of thermodynamics [1]. This analogy is now widely accepted as an actual physical relation rather than just a mathematical anomaly. This in large part is due to Hawking's landmark discovery that black holes radiate thermally [2]. One of the more important open issues in this regard is the microscopic source of black hole entropy [3].

In the case of non-extremal black holes, the quantifying relations for the temperature and entropy are well established: $T=\kappa/2\pi$ and $S=A/4$, respectively, where κ denotes the surface gravity, A denotes the surface area of the outer horizon and all fundamental constants have been set to unity. However, when extremal black holes are considered (i.e., charged or spinning black holes with a degenerate horizon singularity), it is an entirely different matter. There is still no consensus regarding extremal thermodynamics.

Extremal black holes are often interpreted as a limiting case of non-extremal solutions [4], and this viewpoint leads to $T_{ext}=0$ and $S_{ext}=A_{ext}/4>0$. However, Hawking *et al.* [5], Teitelboim [6] and others [7,8] have argued strongly against this intuitive notion. The ''Hawking-Teitelboim conjecture''—that extremal and non-extremal black holes are qualitatively distinct objects—has profound influences on thermodynamics. For instance, it has been argued that T_{ext} is an arbitrary quantity. Quantitatively, this can be explained by the double zero in the extremal metric at the horizon, which translates to no conical singularity in the Euclidean (i.e., imaginary time) sector. Hence, there is no method for fixing the Euclidean time periodicity (which is equivalent to the inverse black hole temperature), contrary to the non-extremal case [9]. Qualitatively, the arbitrary temperature can be interpreted as a consequence of the third law of thermodynamics (i.e., no system with a finite temperature can ever reach $T=0$), which prevents non-extremal black holes from evolving into extremal ones and *vice versa*. An extremal black hole must be free to radiate at any temperature so as to retain its extremal nature, regardless of incoming radiation.

It has been further proposed that the arbitrary temperature implies a vanishing entropy. This argument is based on qualitative differences (in the Euclidean sector) between the extremal and non-extremal topologies. The Euclidean topology of an extremal black hole is relatively trivial, and this effectively eliminates the usual horizon contribution (which accounts for the non-extremal entropy [9]) to the Euclidean action. Note that the findings of various other works have since supported the Hawking-Teitelboim conjecture [10–15].

In spite of the compelling nature of the above arguments, there is still significant opposition to this point of view. Trivedi [16] and Loranz *et al.* [17] have argued that stress tensor regularity on the horizon (in the free-falling observer frame) will be violated unless $T_{ext}=0$. Meanwhile, the strongest case against $S_{ext}=0$ has come from the calculations of Strominger and others [18]. They considered certain classes of weakly coupled string theory (for which massive string states can be represented by extremal black holes) and used a statistical procedure to generate $S_{ext}=A_{ext}/4$, precisely. The same result has been obtained elsewhere with arguments that favor a well-defined extremal limit. These include Ghosh and Mitra [19] and Kiefer and Louko [20] (quantizing the system before extremizing), as well as Zaslavskii [21] (confining the black hole to a finite cavity before extremizing). In still another viewpoint, Wang *et al.* [22] have proposed that distinct extremal solutions (''Hawking's'' and ''Zaslavskii's'') can coexist in nature.

In this paper, we hope to gain further understanding into

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the thermodynamics of extremal black holes. The vehicle for our investigations is a special dilaton model of gravity that describes the $(1+1)$-dimensional projection of a rotating Banados-Teitelboim-Zanelli (BTZ) black hole. The BTZ black hole refers to solutions of $(2+1)$-dimensional anti–de Sitter gravity that were first documented by Banados, Teitelboim and Zanelli [23]. The reduction process to a dilaton model is based on the work of Achucarro and Ortiz [24]. The BTZ model has sparked recent interest due to its profound connections with string theory. That is, many of the black holes pertaining to string theory have near-horizon geometries that can be expressed as BTZ $\times$ simple manifold [25].

Recently, it has been pointed out that the procedures of dimensional reduction and quantization do not necessarily commute [26]. Moreover, there is a so-called ''dimensional-reduction anomaly'' [27] which implies that renormalized quantities in the unreduced theory cannot be simply obtained by renormalizing and summing their dimensionally reduced counterparts. Our interest in the present paper, however, is not necessarily to derive quantitatively accurate corrections to the BTZ black hole thermodynamics. It is to understand qualitatively the difference between extremal and non-extremal geometries. In this regard, the quantization of the dimensionally reduced theory may be adequate. In any case, in the context of $(1+1)$-dimensional dilaton gravity, the dimensional-reduction anomaly can be considered as inconsequential.

The remainder of the paper proceeds as follows. In Sec. II, we consider the non-extremal geometry, including the calculation of back-reaction effects to the first perturbative order. Section III continues the non-extremal study with the evaluation of one-loop thermodynamics by way of a Euclidean action approach [9,28,29]. In Sec. IV, we investigate the extremal limit by applying a limiting procedure to the results of the prior two sections. Section V considers an alternative method for extremal calculations that reflects the topological differences between the extremal and non-extremal geometries. This method is similar to the approach taken by Buric and Radovanovic [13] in the context of Reissner-Nordstrom black holes. Section VI contains a summary of our findings along with a brief discussion.

Note that all calculations are with respect to the Hartle-Hawking vacuum state [30]. This state can be regarded as describing an eternal black hole in thermal equilibrium or (effectively) a black hole within a thermally reflective ''box.''

Although this Introduction has emphasized extremal black holes, the non-extremal results have merit on their own. With this in mind, we make note of other studies [31] that have considered the one-loop corrected thermodynamics of the BTZ black hole.

II. NON-EXTREMAL GEOMETRY

Before proceeding on with the formal discussion, we note that the analytical techniques of this paper are based on a previous one-loop study of generic dilaton gravity [32]. Since the current treatment goes rather quickly over some of the steps, the reader is referred to the above citation for a more detailed discussion.

The initiating point of our formalism is $(2+1)$-dimensional anti–de Sitter gravity. Along with the classical action, we include a matter action that describes minimally coupled, massless, quantized scalar fields. The complete action functional (up to surface terms) can be written as follows:

$$I^{(3)}=\frac{1}{16\pi G^{(3)}}\int d^3x\sqrt{-g^{(3)}}\left(R^{(3)}+\frac{2}{l^2}\right)-\frac{\mathcal{K}}{16\pi G^{(3)}}\int d^3x\sqrt{-g^{(3)}}\sum_{i=1}^{N}(\nabla^{(3)}f_i)^2, \qquad (1)$$

where f_i denotes the matter fields, N is a large integer, $G^{(3)}$ is the $3D$ Newton constant, $-2l^{-2}$ is the negative cosmological constant and $\mathcal{K}$ is a coupling parameter that vanishes in the classical limit (i.e., as $\hbar\to 0$).

Axial symmetry can be imposed on this action by way of the following metric ansatz [24]:

$$ds^{2(3)}=g_{\mu\nu}dx^{\mu}dx^{\nu}+\phi^2(\alpha d\theta+A_{\mu}dx^{\mu})^2, \qquad (2)$$

where $\mu,\nu=\{0,1\}$, α is an arbitrary constant of dimension length, A_μ is a vector gauge field, ϕ is a scalar field (the ''dilaton'') and all fields are functions of only $\{x^0,x^1\}=\{t,x\}$. This reduction process results in the following $(1+1)$-dilaton model:

$$I=\int d^2x\sqrt{-g}\,\phi\left(R+2l^{-2}-\frac{1}{4}\phi^2F^{\mu\nu}F_{\mu\nu}\right)-\mathcal{K}\int d^2x\sqrt{-g}\,\phi\sum_{i=1}^{N}(\nabla f_i)^2, \qquad (3)$$

where we have set $\alpha=8G^{(3)}$ without loss of generality. The ''field-strength'' tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is known to be directly related to the angular momentum of a rotating BTZ black hole [24]. Note that the reduced action describes constant curvature gravity with coupling to both an Abelian gauge field and conformally invariant matter fields.

Since the action is ultimately significant as the exponent in a path integral, it is possible to ''integrate out'' the matter fields and then consider the vacuum limit (i.e., $f_i\to 0$). In this event, the resultant ''effective action'' can be (at least) partially derived in conformally invariant matter theories because of its exploitable relation to the trace conformal anomaly [33]. For the special case of conformally invariant matter in two spacetime dimensions, the effective action can be derived exactly up to terms that are conformally invariant [34]. For the dilaton model of interest, we find (assuming that $N\gg 1$ and the black hole is massive when compared to the Planck scale)

$$I=I_{CL}-\mathcal{K}\int d^2x\sqrt{-g}\left[R\frac{1}{\Box}R-\frac{3}{\phi^2}(\nabla\phi)^2\left(\frac{1}{\Box}R-\ln\mu^2\right)\right.$$
$$\left.-6\ln(\phi)R\right], \tag{4}$$

where I_{CL} is the left-most integral in Eq. (3), $\mathcal{K}$ has been appropriately rescaled (now, $\mathcal{K}\approx N\hbar$) and μ is an arbitrary parameter that arises out of renormalization procedures [33]. The precise forms of the functional coefficients (in this case, $3/\phi^2$ and $6\ln\phi$) depend upon the form of dilaton-matter coupling that arises out of the reduction process.

It should be pointed out that the conformally invariant portion of the effective action, which is described by the $\ln\mu^2$ term in Eq. (4), is incomplete as shown. This portion cannot be found in a closed form, but it can be approximated by an expansion (in powers of curvature) of which we have only included the leading-order term [35]. Recently, non-local terms of this expansion, which appear to be relevant to the perturbative order of Eq. (4), have been calculated [36]. Because of their non-local nature, the incorporation of such terms into our formalism is by no means a straightforward process. Consequently, for the sake of simplicity, we have omitted these terms in the current analysis. This issue is further addressed in the final section.

It is convenient to re-express the effective action in an equivalent local form as follows:

$$I=I_{CL}-\mathcal{K}\int d^2x\sqrt{-g}\left[(\psi+\chi)R+g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\chi\right.$$
$$\left.-\frac{3}{\phi^2}(\nabla\phi)^2(\psi-\ln\mu^2)-6\ln(\phi)R\right], \tag{5}$$

where ψ and χ are a pair of auxiliary scalar fields[1] that are constrained according to the following equations:

$$\Box\psi=R, \tag{6}$$

$$\Box\chi=R-\frac{3}{\phi^2}(\nabla\phi)^2. \tag{7}$$

By varying the effective action (5) with respect to the metric, dilaton and Abelian gauge field, we obtain

$$-2\nabla_\mu\nabla_\nu\phi+2g_{\mu\nu}\Box\phi-\frac{2}{l^2}g_{\mu\nu}\phi-\frac{1}{4}(3g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$$
$$-4g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta})\phi^3=T_{\mu\nu}, \tag{8}$$

$$R+\frac{2}{l^2}-\frac{3}{4}\phi^2F^{\mu\nu}F_{\mu\nu}=D, \tag{9}$$

$$\nabla_\mu(F^{\mu\nu}\phi^3)=0, \tag{10}$$

respectively, where

$$T_{\mu\nu}\equiv-\mathcal{K}\Big[2\nabla_\mu\nabla_\nu(\psi-6\ln(\phi)+\chi)-(\nabla_\mu\chi\nabla_\nu\psi$$
$$+\nabla_\mu\psi\nabla_\nu\chi)-g_{\mu\nu}\{2\Box[\psi-6\ln(\phi)+\chi]$$
$$-g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\psi\}+\frac{3}{\phi^2}(\psi-\ln\mu^2)[2\nabla_\mu\phi\nabla_\nu\phi$$
$$-g_{\mu\nu}(\nabla\phi)^2]\Big], \tag{11}$$

$$D\equiv6\mathcal{K}\{\phi^{-3}(\nabla\phi)^2(\psi-\ln\mu^2)+g^{\mu\nu}\nabla_\mu[\phi^{-2}(\psi$$
$$-\ln\mu^2)\nabla_\nu\phi]-\phi^{-1}R\}. \tag{12}$$

Note that $T_{\mu\nu}$ can be identified with the quantum stress tensor.

The ''Maxwell'' field equation (10) can be trivially integrated to yield

$$F^{\mu\nu}=\frac{1}{l}\frac{\epsilon^{\mu\nu}}{\sqrt{-g}}\frac{J}{\phi^3}, \tag{13}$$

where $\epsilon^{\mu\nu}$ is the Levi-Cività symbol and J is an integration constant that can be identified with the Abelian charge observable (i.e., quantized angular momentum). The above result inspires the definition of an ''effective potential'' $V_J(\phi)\equiv l^{-2}(2\phi-\frac{1}{2}J^2\phi^{-3})$, which leads to the remaining field equations (8,9) taking on the following compact forms:

$$-2\nabla_\mu\nabla_\nu\phi+2g_{\mu\nu}\Box\phi-g_{\mu\nu}V_J(\phi)=T_{\mu\nu}, \tag{14}$$

$$R+\frac{\partial V_J}{\partial\phi}=D. \tag{15}$$

It is instructive to first consider the classical ($\mathcal{K}=0$) solution. A prior work has demonstrated how to obtain the classical solution in a static gauge for a wide class of dilaton models [38]. For reduced BTZ gravity, this solution can be expressed as follows:

$$\phi_{CL}=\frac{x}{l}, \tag{16}$$

$$(A_t)_{CL}=-\frac{l^2J}{2x^2}, \tag{17}$$

$$ds^2=-g_{CL}(x)dt^2+g_{CL}^{-1}(x)dx^2, \tag{18}$$

$$g_{CL}(x)=\frac{x^2}{l^2}-lM+\frac{J^2l^2}{4x^2}, \tag{19}$$

where we have assumed (without loss of generality) a time-like gauge vector and M is a constant parameter that can be identified with the Arnowitt-Deser-Misner (ADM) mass observable. It is useful to note that $R_{CL}=-g''_{CL}$ (where primes

[1]Auxiliary fields of an analogous form were first used in Ref. [37] in the context of spherically symmetric gravity.

indicate differentiation with respect to x) and $g_{CL}(x) = -|k^{\mu}|^2$, where $k^{\mu} = l(\sqrt{-g})^{-1}\epsilon^{\mu\nu}\partial_{\nu}\phi$ is a Killing vector for the classical field equations.

For subsequent calculations, it is convenient to re-express $g_{CL}(x)$ in the following form:

$$g_{CL}(x) = \frac{1}{l^2x^2}(x^2 - x_o^2)(x^2 - x_i^2), \tag{20}$$

where

$$x_o^2 = \frac{l^3}{2}[M + \sqrt{M^2 - J^2/l^2}], \tag{21}$$

$$x_i^2 = \frac{l^3}{2}[M - \sqrt{M^2 - J^2/l^2}]. \tag{22}$$

The positive root of x_o^2/x_i^2 locates the classical outer/inner event horizon. Since we have restricted considerations to black hole solutions (and non-extremal ones until Sec. IV), the phase space of observables is restricted by $M>0$ and $M^2 > J^2/l^2$.

For a higher-order analysis, it is necessary to introduce a suitable ansatz for describing the back-reaction effects on the classical geometry. Following a proposal by Frolov *et al.* [29], we now express the quantum-corrected solution in the following manner:

$$\phi = \phi_{CL} = x/l, \tag{23}$$

$$ds^2 = -e^{2\omega(x)}g(x)dt^2 + g^{-1}(x)dx^2, \tag{24}$$

$$g(x) = g_{CL}(x) - lm(x), \tag{25}$$

where the fields $m(x)$ and $\omega(x)$ must vanish as $\mathcal{K} \to 0$. Note that $A_t = (A_t)_{CL}$ follows trivially, since we have assumed no coupling between the matter and Abelian sectors.

By substituting the above ansatz into the field equations, we find that Eq. (15) and the off-diagonal component of Eq. (14) are both identically vanishing. After some simplification, the ''surviving'' field equations are found to be

$$-e^{2\omega}gm' = T_{tt}, \tag{26}$$

$$-\frac{m'}{g} + \frac{2}{l}\omega' = T_{xx}. \tag{27}$$

If these expressions are truncated at the one-loop level (i.e., at first order in $\mathcal{K}$), then we obtain the elegant results

$$m' = -T_t^t, \tag{28}$$

$$\omega' = \frac{l}{2g_{CL}}(T_x^x - T_t^t). \tag{29}$$

Next in this study, we explicitly formulate the auxiliary fields $\psi(x)$ and $\chi(x)$. Since we are ultimately deriving one-loop expressions, it is sufficient to express these fields in terms of the classical geometry. Furthermore, the choice of boundary conditions should reflect the Hartle-Hawking vacuum state [30]. Such conditions restrict the analysis to solutions that are periodic in Euclidean (i.e., imaginary) time when on a spatial manifold extending from the outer horizon to a fixed outer boundary L [9].

Let us first consider solving Eq. (6) for ψ. The appropriate solution can be found by way of a special map [29]: the classical Euclidean geometry (in a static gauge) conformally mapped to the geometry of a ''disk.'' Significantly, the disk geometry can be interpreted as the Rindler coordinate description of the Hartle-Hawking state for a flat spacetime [33]. On the basis of Eqs. (6),(18), such a map can be suitably described by

$$g_{CL}(x)(idt)^2 + g_{CL}^{-1}(x)dx^2 = e^{-\psi(z)}[z^2d\theta^2 + dz^2], \tag{30}$$

where the disk coordinates are confined to $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq L_z$. Solving for $\psi(z(x))$, we find

$$\psi(x) = -\ln g_{CL}(x) - \frac{4\pi}{\beta_{CL}}\int_x^L \frac{dx}{g_{CL}(x)} - 2\ln\left(\frac{\beta_{CL}}{2\pi L_z}\right), \tag{31}$$

where β_{CL} denotes the Euclidean time periodicity for the classical system (i.e., $0 \leq it \leq \beta_{CL}$).

We can determine $\chi(x)$ by integrating Eq. (7) and then imposing the constraint that $\chi \to \psi$ in the limit of minimal dilaton-matter coupling (for which the effective action assumes a ''Polyakov-like'' form [39]). This procedure leads to[2]

$$\chi(x) = \psi(x) + 3\int_x^L \frac{dx}{g_{CL}(x)}\int_{x_o}^x \frac{d\tilde{x}g_{CL}(\tilde{x})}{\tilde{x}^2}. \tag{32}$$

By substituting the classical solution (16),(18),(31),(32) into the stress tensor (11), we obtain the following one-loop expressions:

$$T_t^t = \frac{\mathcal{K}}{g_{CL}}\Bigg[(g'_{CL})^2 - 4g_{CL}g''_{CL} - \frac{16\pi^2}{\beta_{CL}^2} + 6\frac{g_{CL}}{x^2}(2g_{CL} - xg'_{CL}) - 3\frac{g_{CL}^2}{x^2}\left(2 + \ln g_{CL} + \frac{4\pi}{\beta_{CL}}\int_x^L \frac{dx}{g_{CL}} + \ln \Upsilon^2\right) + \frac{12\pi}{\beta_{CL}}\int_{x_0}^x \frac{dxg_{CL}}{x^2}\Bigg], \tag{33}$$

$$T_x^x = \frac{\mathcal{K}}{g_{CL}}\Bigg[\frac{16\pi^2}{\beta_{CL}^2} - (g'_{CL})^2 - \frac{6}{x}g_{CL}g'_{CL} + 3\frac{g_{CL}^2}{x^2}\left(\ln g_{CL} + \frac{4\pi}{\beta_{CL}}\int_x^L \frac{dx}{g_{CL}} + \ln \Upsilon^2\right) - \frac{12\pi}{\beta_{CL}}\int_{x_0}^x \frac{dxg_{CL}}{x^2}\Bigg], \tag{34}$$

[2]The $x = x_o$ integration limit (besides being an intuitive choice) can be uniquely fixed by constraining the curvature R to be regular on the horizon.

where $\Upsilon \equiv \mu \beta_{CL}/2\pi L_z$ can be regarded as an arbitrary parameter.

When on the constraint surface, the Euclidean time periodicity must be suitably fixed to ensure the horizon regularity of the Euclidean geometry (i.e., to eliminate any conical singularity or deficit angle) [9]. On this basis, we can explicitly evaluate the on-shell value of β_{CL} by matching the classical solution with a conical geometry,

$$g_{CL}(x)(idt)^2+g_{CL}^{-1}(x)dx^2=z^2d\theta^2+H(z)dz^2 \tag{35}$$

(where $0\leqslant\theta\leqslant2\pi$ and $z=0$ at $x=x_o$), and then enforcing $H(0)=1$. This process yields

$$\beta_{CL}=\frac{4\pi}{g'_{CL}}\bigg|_{x=x_o}=\frac{2\pi l^2 x_o}{x_o^2-x_i^2}. \tag{36}$$

By substituting Eq. (33) into Eq. (28), integrating and also incorporating Eqs. (20),(36), we find

$$m(x)=2\frac{\mathcal{K}}{l^2}\Bigg[2x-3\frac{x_o^2+x_i^2}{x}-8x_i\ln\left(\frac{x-x_i}{x+x_i}\right)+\frac{1}{2}\left(3x+3\frac{x_o^2+x_i^2}{x}-\frac{x_o^2x_i^2}{x^3}\right)\left(\frac{x_i}{x_o}\ln\left(\frac{x-x_i}{x+x_i}\right)+\ln\left(\frac{(x+x_o)^2(x^2-x_i^2)}{l^2x^2}\right)+\Theta\right)\Bigg]+m_0, \tag{37}$$

where

$$\Theta\equiv-\frac{x_i}{x_o}\ln\left(\frac{L-x_i}{L+x_i}\right)+\ln\left(\frac{L-x_o}{L+x_o}\right)+\ln\Upsilon^2 \tag{38}$$

and m_0 is an integration constant that can be absorbed (without loss of generality) into the classical mass M. Next, let us invoke the convention $\omega(L)=0$ and define a function $\varpi(x)$ in accordance with $\omega(x)=\mathcal{K}l[\varpi(L)-\varpi(x)]$. Then the substitution of Eqs. (33),(34) into Eq. (29) ultimately yields

$$\begin{aligned}\varpi(x)=&-\frac{1}{x}+2\frac{3x_o^2+x_i^2}{(x_o^2-x_i^2)(x+x_o)}-2\frac{x_i^2(x_o^2+3x_i^2)+x_o(x_o^2-5x_i^2)x}{x_o(x_o^2-x_i^2)(x^2-x_i^2)}\\&-8\frac{x_i}{(x_o+x_i)^2}\ln\left(\frac{x-x_i}{x+x_o}\right)+8\frac{x_i}{(x_o-x_i)^2}\ln\left(\frac{x+x_i}{x+x_o}\right)\\&+\frac{3}{x}\left[\frac{x_i}{x_o}\ln\left(\frac{x-x_i}{x+x_i}\right)+\ln\left(\frac{(x+x_o)^2(x^2-x_i^2)}{l^2x^2}\right)+\Theta\right].\end{aligned} \tag{39}$$

Note that $m(x)$ and $\omega(x)$ are both well-defined functions for $x_o\leqslant x\leqslant L$, thereby substantiating our choice of ansatz.

We next consider the quantum-corrected curvature. This can be written as $R=-e^{-\omega}[e^{-\omega}(e^{2\omega}g)']'$ or, for a one-loop truncation,

$$R=-g''_{CL}+lm''-2\omega''g_{CL}-3\omega'g'_{CL}. \tag{40}$$

Substituting the prior results for $m(x)$ and $\omega(x)$, evaluating the derivatives and then simplifying, we obtain

$$\begin{aligned}R=&-\frac{2}{x^4l^2}(x^4+3x_o^2x_i^2)+\frac{2\mathcal{K}}{lx}\Bigg\{24+\frac{6}{x_ox}(x_o^2-x_i^2)+\frac{4}{x_ox^4(x+x_o)^2(x^2-x_i^2)^2}[x_i^6x_o^2(3x^3+6x_ox^2+8x_o^2x+4x_o^3)\\&-x_i^4x^2(3x^5+6x_ox^4+5x_o^2x^3+8x_o^3x^2+11x_o^4x+6x_o^5)-x_i^2x^5(3x^4-4x_ox^3-11x_o^2x^2-6x_o^3x+3x_o^4)+3x_o^2x^9]\\&+\frac{3}{x^4}[3x^4-(x_o^2+x_i^2)x^2-x_o^2x_i^2]\left[\frac{x_i}{x_o}\ln\left(\frac{x-x_i}{x+x_i}\right)+\ln\left(\frac{(x+x_o)^2(x^2-x_i^2)}{l^2x^2}\right)+\Theta\right]\Bigg\},\end{aligned} \tag{41}$$

which is also a well-defined quantity throughout the relevant manifold.

Let us next consider the one-loop shift in the outer horizon Δx_o. To determine this shift, we begin with a first-order Taylor expansion of the function $g(x_o+\Delta x_o)$; cf. Eq. (25). After expanding and applying the horizon conditions $g_{CL}(x_o)=g(x_o+\Delta x_o)=0$ (with the latter valid being valid to first order), we find

$$\Delta x_o=\frac{l^3x_o}{2(x_o^2-x_i^2)}m(x_o)=\frac{l\beta_{CL}}{4\pi}m(x_o). \tag{42}$$

Note that a similar calculation is not viable at the inner horizon, since the back-reaction ansatz has not been strictly defined for $x<x_o$. Furthermore, the shift in x_i is expected to be non-analytic in $\mathcal{K}$ [33].

III. NON-EXTREMAL THERMODYNAMICS

Our method of thermodynamic analysis is based on the well-known techniques of Gibbons and Hawking [9] and others [28,29]. This procedure can be summarized as follows. After analytically continuing to Euclidean spacetime and closing off the imaginary time direction, one finds that the path integral can be interpreted as a thermodynamic partition function $\mathcal{Z}$. This partition function describes an ensemble of black holes that are radiating at a temperature β^{-1}, where β corresponds to the periodicity of Euclidean time. Furthermore, a semi-classical approximation has been shown to yield the relation [9]

$$\ln(\mathcal{Z})=-I_{OS}, \tag{43}$$

where I_{OS} denotes the on-shell Euclidean action. Note for an on-shell system that β^{-1} corresponds to the so-called ''Hawking temperature'' of black hole radiation.[3]

Let us reconsider the effective action of Eq. (5). By transforming to Euclidean spacetime (i.e., rotating $t\to it$ and re-expressing all geometrical objects in terms of a positive-definite metric[4]) and also applying the static solution of Eqs. (23),(24), we obtain the following Euclidean form of the action:

$$I=-\beta\int_{x_q}^{L}dxe^{\omega}\left(\frac{x}{l}R+V_J\left(\frac{x}{l}\right)-\mathcal{K}\left\{\left[\psi-6\ln\left(\frac{x}{l}\right)+\chi\right]R+g\chi'\psi'-3\frac{g}{x^2}(\psi-\ln\mu^2)\right\}\right)+\beta\left(\int^{L}dxe^{\omega}R\right)\times\left\{\frac{x}{l}-\mathcal{K}\left[\psi-6\ln\left(\frac{x}{l}\right)+\chi\right]\right\}\bigg|_{x=L}-\frac{\beta J}{l}\Delta A_t, \tag{44}$$

where x_q represents the quantum-corrected outer horizon, $\Delta A_t\equiv[A_t(L)-A_t(x_q)]$ and note that $e^{\omega}R$ is a total derivative. So as to ensure a well-defined variational principle at the boundaries of the system, we have included the appropriate surface terms in the third line of this expression. Except for the right-most (charge sector) term, this surface contribution is directly analogous to Gibbons and Hawking's ''extrinsic curvature term'' [9].[5]

The Euclidean action can be written in a more convenient form by way of Eq. (14). Let us first define $G_{\mu\nu}$ as the left-hand side of this field equation and then express both G_{tt} and T_{tt} [which can be obtained[6] from Eq. (11)] in terms of the static solution. After integrating the curvature terms in the Euclidean action (44) by parts, we can incorporate the static forms of G_{tt} and T_{tt} to obtain

$$I=\beta\int_{x_q}^{L}dx\left(e^{\omega}(G_t^t-T_t^t)-\left\{\frac{2}{l}e^{\omega}g+4\mathcal{K}e^{-\omega}(e^{2\omega}g)'+2\mathcal{K}e^{\omega}g\left[\psi+6\ln\left(\frac{x}{l}\right)-\chi\right]'\right\}'\right)+\beta\left(\int^{x_q}dxe^{\omega}R\right)\times\left\{\frac{x}{l}-\mathcal{K}\left[\psi-6\ln\left(\frac{x}{l}\right)+\chi\right]\right\}\bigg|_{x=x_q}-\frac{\beta J}{l}\Delta A_t. \tag{45}$$

Evidently, the above integrand vanishes on the constraint surface up to a total divergence. It follows that the on-shell Euclidean action reduces to just a surface expression, and this is found to be

$$I_{OS}=-\beta\left\{\frac{2}{l}g+4\mathcal{K}(e^{2\omega}g)'+2\mathcal{K}g\left[\psi+6\ln\left(\frac{x}{l}\right)-\chi\right]'\right\}\bigg|_{x=L}-4\pi\left\{\frac{x}{l}-\mathcal{K}\left[\psi-6\ln\left(\frac{x}{l}\right)+\chi\right]\right\}\bigg|_{x=x_q}-\frac{\beta J}{l}\Delta A_t. \tag{46}$$

Here, we have used $\omega(L)=g(x_q)=0$ and the perturbative analogue of Eq. (36):

$$\beta=4\pi\frac{e^{-\omega}}{g'}\bigg|_{x=x_q}. \tag{47}$$

We now recall the relation $\ln(\mathcal{Z})=-I_{OS}$ and point out that (on the basis of thermodynamic arguments) the logarithm of the partition function should ultimately take on the following free energy form:

$$\ln(\mathcal{Z})=-\beta_L\left[E-\sum_{\eta}\eta\gamma_\eta\right]+S, \tag{48}$$

where β_L is the fixed value of the inverse temperature at the outer boundary of the system, E is the thermal energy, η is an intrinsically conserved quantity, γ_η is the related chemical potential and S is the entropy of the system. By comparing the two expressions for $\ln\mathcal{Z}$, we are able to make the following identifications:

---

[3]Keep in mind that an observer at x locally measures an inverse temperature of $\sqrt{-g_{tt}[x]}\beta$, that is, a ''red-shifted'' value of inverse temperature [28]. For anti–de Sitter spacetimes (unlike for asymptotically flat ones), this red-shift factor diverges as $x\to\infty$.

[4]Technically, the Abelian charge should also be complexified so that $A_t dt$ remains invariant [28]. It is implied, however, that we have already continued back to a real charge before presenting any result in this paper.

[5]Technically, we should also include an analogous horizon term, as well as a delta-function contribution from the curvature [40]. However, these horizon contributions are known to ultimately cancel in the final on-shell expressions [29], and hence are not pertinent to the thermodynamics.

[6]It is helpful to first make the substitution $\Box(\psi+\chi)=\Box[2\psi-3\phi^{-2}(\nabla\phi)^2]$; cf. Eqs. (6),(7).

$$E=-2\frac{\beta}{\beta_L}\left\{\frac{1}{l}g_{CL}-m+2\mathcal{K}g'_{CL}+\mathcal{K}g_{CL}\left[\psi+6\ln\left(\frac{x}{l}\right)-\chi\right]'\right\}\Bigg|_{x=L}, \tag{49}$$

$$S=\frac{4\pi}{l}(x_o+\Delta x_o)-4\pi\mathcal{K}\left[\psi-6\ln\left(\frac{x}{l}\right)+\chi\right]\Bigg|_{x=x_o}, \tag{50}$$

$$\gamma_J=\frac{1}{l}\frac{\beta}{\beta_L}[A_t(L)-A_t(x_o+\Delta x_o)]. \tag{51}$$

Before any further evaluation, two points should be clarified:

(i) The inverse boundary temperature β_L is ''red-shifted'' from the inverse Hawking temperature β according to [28] $\beta_L=\sqrt{g_{tt}[x=L]}\beta=\sqrt{g(L)}\beta$.

(ii) The Euclidean action is known to diverge as the outer boundary tends to infinity [9], which implies that the calculated energy will also diverge unless a suitable subtraction procedure is invoked. The usual convention is to subtract off the energy contribution from the asymptotic geometry [28], and so we define a subtracted energy according to $E_{sub}\equiv E[g(L)]-E[g_\infty]$, where $g_\infty\equiv L^2/l^2$.

By substituting the prior geometrical formalism into Eqs. (47),(49)–(51) and also using binomial expansions where applicable, we obtain the following one-loop expressions:

$$T\equiv\beta^{-1}=\beta_{CL}^{-1}+\mathcal{K}l\beta_{CL}^{-1}\left\{-\frac{9x_o^4+6x_o^2x_i^2+x_i^4}{(x_o^2-x_i^2)^2x_o}+16\frac{x_ox_i^2}{(x_o^2-x_i^2)^2}\left[2\ln\left(\frac{l}{x_o}\right)+\Theta\right]+\varpi(L)\right\}, \tag{52}$$

$$E_{sub}=2\frac{L}{l^2}\left(1-\frac{1}{L^2}\sqrt{(L^2-x_o^2)(L^2-x_i^2)}\right)+\frac{lLm(L)}{\sqrt{(L^2-x_o^2)(L^2-x_i^2)}}+2\frac{\mathcal{K}}{l}\left[13-\frac{3x_o}{L}-\frac{13x_oL^4-2(3x_o^2+x_i^2)L^3-3x_o(x_o^2+x_i^2)L^2+x_o^3x_i^2}{x_oL^2\sqrt{(L^2-x_o^2)(L^2-x_i^2)}}\right], \tag{53}$$

$$S=\frac{4\pi x_o}{l}-\frac{4\pi\mathcal{K}}{x_o^2-x_i^2}\left\{x_o^2+3x_i^2+\frac{x_i}{x_o}(3x_o^2+x_i^2)\ln\left(\frac{L-x_i}{L+x_i}\right)+2(3x_o^2+x_i^2)\ln\left(\frac{L+x_o}{x_o}\right)-(x_o^2+3x_i^2)\ln\left(\frac{L^2-x_i^2}{x_o^2-x_i^2}\right)-(x_o^2-x_i^2)\ln\left(\frac{x_o^5L}{l^6}\right)-(5x_o^2-x_i^2)\left[\ln\left(\frac{x_o^2-x_i^2}{l^2}\right)+\Theta\right]\right\}, \tag{54}$$

$$\gamma_J=\frac{l^2J(L^2-x_o^2)}{2x_o^2L\sqrt{(L^2-x_o^2)(L^2-x_i^2)}}\left[1+\frac{1}{2}\frac{l^3L^2}{L^2-x_o^2}\left(\frac{m(L)}{L^2-x_i^2}-\frac{2m(x_o)}{x_o^2-x_i^2}\right)\right]. \tag{55}$$

A brief comment regarding the one-loop entropy is in order. Although the black hole entropy is normally a property of the horizon, the above expression (54) contains terms that depend on the ''box size'' L. This paradoxical behavior can be attributed to the non-local nature of the auxiliary fields ψ and χ; cf. Eqs. (6),(7). Even at the horizon, these fields contain information with regard to the entire manifold. Physically, the L-dependent terms can be attributed to a ''hot thermal gas'' that fills up the box.

For a check on validity, it is helpful to consider the classical limit. First, we can re-express the classical entropy in the usual ''Bekenstein-Hawking'' form (i.e., $S=A/4G^{(3)}$) by making the following identification [cf. Eq. (2)]: $A=16\pi G^{(3)}\phi(x_o)$ is the circumference of the outer horizon. Let us next consider the behavior of the classical energy in the $L\to\infty$ limit. Under these conditions, it can be shown that $\sqrt{g(L)}E_{sub}\to M$, which is the expected asymptotic behavior of a quasi-localized energy in an anti–de Sitter spacetime [41]. A similar analysis for the chemical potential yields the limit $\sqrt{g(L)}\gamma_J\to lJ/2x_o^2$, which is the form of the rotational potential that might be anticipated for an axially symmetric system of radius x_o and angular momentum J. Finally, it can be readily verified [4] that the classical limit of T satisfies the expected relation between the Hawking temperature and the surface gravity: i.e., $T=\kappa/2\pi$.

A final thermodynamic consideration is the flux of thermal radiation. This flux has both an emission and absorption component that are equal in magnitude (assuming the Hartle-Hawking state). Furthermore, it has been shown [42] that the flux components are equivalent to the diagonal components of the stress tensor if these tensor components are expressed in terms of suitably defined null coordinates. In regard to the classical BTZ geometry, the appropriate coordinates can be defined as follows:

$$u=t-\int\frac{dx}{g_{CL}},\quad v=t+\int\frac{dx}{g_{CL}}. \tag{56}$$

It can be readily shown that

$$T_{uu}=T_{vv}=-\frac{g_{CL}}{4}(T_t^t-T_x^x). \tag{57}$$

Note that T_{uu}/T_{vv} represents the outgoing/incoming flux and T_{uv} can be obtained by ''flipping'' the sign in front of T_x^x.

By incorporating Eqs. (33),(34) into the above relation, we find the following results:

$$\begin{aligned}T_{uu}=&-\frac{\mathcal{K}}{2}\frac{(x-x_o)^2}{l^4x_o^2x^6}\Bigg\{3x_o^2x^6+6x_o(3x_o^2-2x_i^2)x^5\\&+(3x_o^4-2x_o^2x_i^2+4x_i^4)x^4+4x_ox_i^2(2x_o^2-x_i^2)x^3\\&+3x_o^2x_i^2(2x_o^2-3x_i^2)x^2-10x_o^3x_i^4x-5x_o^4x_i^4\\&-3x_o^2(x+x_o)^2(x^2-x_i^2)^2\Bigg[\frac{x_i}{x_o}\ln\left(\frac{x-x_i}{x+x_i}\right)\\&+\ln\left(\frac{(x+x_o)^2(x^2-x_i^2)}{l^2x^2}\right)+\Theta\Bigg]\Bigg\},\end{aligned} \tag{58}$$

$$\begin{aligned}T_{uv}=&\frac{\mathcal{K}}{2}\frac{(x^2-x_o^2)(x^2-x_i^2)}{l^4x^6}\\&\times[13x^4+3(x_o^2+x_i^2)x^2-3x_o^2x_i^2].\end{aligned} \tag{59}$$

Note the divergence of these components as $x\to\infty$. An asymptotically divergent flux is an expected outcome for an anti–de Sitter theory. Since an asymptotic observer locally measures a vanishing temperature (see footnote 3), it follows that she would detect an infinite flux of particles.

The above calculations provide a further check on our formalism. Christensen and Fulling [43] have shown that enforcing stress tensor regularity at the outer horizon in the free-falling frame (which is a necessary condition for describing the Hartle-Hawking state) leads to a certain class of constraints. These translate to the (outer) horizon regularity of the following three quantities:

$$\text{(i) } T_{vv},\quad \text{(ii) } T_{uv}/g_{CL},\quad \text{(iii) } T_{uu}/g_{CL}^2. \tag{60}$$

The above expressions satisfy all three of these constraints by virtue of the $x-x_o$ factor(s) in front.

IV. EXTREMAL LIMIT

It is straightforward to consider the extremal limit of the prior calculations. By definition, the extremal limit corresponds to a coincidence in the classical horizons:[7] $x_i\to x_o$ or $J^2\to l^2M^2$. This limiting procedure leads to the following results:

$$\begin{aligned}m(x)=&2\frac{\mathcal{K}}{l^2}\Bigg\{2x-6\frac{x_o^2}{x}-8x_o\ln\left(\frac{x-x_o}{x+x_o}\right)\\&+\frac{1}{2}\left(3x+\frac{6x_o^2}{x}-\frac{x_o^4}{x^3}\right)\left[2\ln\left(\frac{x^2-x_o^2}{lx}\right)+\ln\Upsilon^2\right]\Bigg\},\end{aligned} \tag{61}$$

$$\varpi(x)\to\text{quadratically divergent throughout the manifold}, \tag{62}$$

$$\begin{aligned}R=&-2\frac{x^4+3x_o^4}{l^2x^4}+\frac{2\mathcal{K}}{lx}\Bigg\{24+4\frac{x_o^2}{x^4}(x^2+x_o^2)\\&+\frac{3}{x^4}(3x^2+x_o^2)(x^2-x_o^2)\left[2\ln\left(\frac{x^2-x_o^2}{lx}\right)+\ln\Upsilon^2\right]\Bigg\},\end{aligned} \tag{63}$$

$$\Delta x_o\to\text{linearly divergent}, \tag{64}$$

$$T=0+\text{linearly divergent corrections}, \tag{65}$$

$$E_{sub}=2\frac{x_o^2}{l^2L}+\frac{lLm(L)}{L^2-x_o^2}+2\mathcal{K}\frac{x_o}{lL^2}\left(\frac{5L^2-2x_oL+x_o^2}{L+x_o}\right), \tag{66}$$

$$S=\frac{4\pi x_o}{l}+\text{linearly divergent corrections}, \tag{67}$$

$$\gamma_J=\frac{l^2J}{2Lx_o^2}+\text{linearly divergent corrections}, \tag{68}$$

$$\begin{aligned}T_{uu}=&-\frac{\mathcal{K}}{2}\frac{(x-x_o)^2}{l^4x^6}\Bigg\{3x^6+6x_ox^5+5x_o^2x^4+4x_o^3x^3\\&-3x_o^4x^2-10x_o^5x-5x_o^6-3(x-x_o)^2\\&\times(x+x_o)^4\left[2\ln\left(\frac{x^2-x_o^2}{lx}\right)+\ln\Upsilon^2\right]\Bigg\},\end{aligned} \tag{69}$$

$$T_{uv}=\frac{\mathcal{K}}{2}\frac{(x-x_o)^2(x+x_o)^2}{l^4x^6}[13x^4+6x_o^2x^2-3x_o^4]. \tag{70}$$

With only a few exceptions (energy, curvature and flux), we find the one-loop results to be poorly defined in the extremal limit. [Note that $m(x)$ has a logarithmic divergence at

[7] By invoking a limiting procedure, it is implied that the extremal condition may be violated by radiation effects. That is, the one-loop corrected horizons may or may not coincide.

the outer horizon.] Furthermore, the stress tensor component T_{uu} fails the previously discussed regularity condition (60),[8] since we now have $g_{CL}\propto(x-x_o)^2$. One must conclude that the one-loop ansatz breaks down in this extremal limiting case.

V. ALTERNATIVE APPROACH TO EXTREMAL CASE

In this section, we reconsider the extremal case by invoking an ansatz (for quantum corrections) that presumes an extremal solution from the beginning. There is ample justification for such a procedure because of topological differences in the extremal and non-extremal solutions [6].

The methodology of this section is to repeat the prior calculations with three fundamental differences:

(i) In place of the classical metric function of Eq. (20), we now use

$$g_{CL}(x)=\frac{1}{l^2x^2}(x^2-x_o^2)^2, \tag{71}$$

where $x_o=\sqrt{l^3M/2}$ and it is useful to remember $g'_{CL}(x_o)=0$. Note that the perturbative ansatz of Eqs. (23)–(25) is otherwise unaltered.

(ii) We now regard the Euclidean time periodicity β as an arbitrary quantity. This proposal is based on the following observation: the extremal (Euclidean) geometry has no conical singularity to be regulated [5].

(iii) In solving for the auxiliary fields ψ and χ, we employ a different method of imposing Hartle-Hawking boundary conditions. First, the associated field equations (6),(7) can be directly integrated to yield

$$\psi(x)=-\ln g_{CL}+\frac{C_\psi}{l^2}\int^x\frac{dx}{g_{CL}}, \tag{72}$$

$$\chi'=-\frac{1}{g_{CL}}\left[g'_{CL}-\frac{C_\chi}{l^2}+3\int^x dx\,\frac{g_{CL}}{x^2}\right], \tag{73}$$

where C_ψ and C_χ are integration constants of dimension length. (Note that the second integration constant in ψ can be absorbed into $\ln\mu^2$ without loss of generality, whereas χ only appears in the formalism as a derivative.) The next step in this method is to constrain the pair of integration constants. For this purpose, we impose (outer) horizon regularity on three geometrical functions: $m(x)$, $\varpi(x)$ and $R(x)$. By evaluating each of these quantities (for arbitrary $C_{\psi,\chi}$) and locating the horizon singularities in the resultant expressions, we are able to identify the following set of constraints:

(a) $m\to C_\psi+C_\chi=8x_o$ or $C_\psi=0$,
(b) $\varpi\to C_\psi^2-\frac{3}{2}C_\psi C_\chi-24C_\psi x_o=-128x_o^2$ and (a),
(c) $R\to 2C_\psi-3C_\chi=16x_o$,

which has a unique solution of

$$C_\psi=8x_o \quad\text{and}\quad C_\chi=0. \tag{74}$$

With regard to (iii), it is worth noting that the same method can be applied to the non-extremal case. By imposing horizon regularity on the non-extremal geometry, we find that $C_\psi=2(x_o^2-x_i^2)/x_o$ and $C_\chi=2(3x_o^2+x_i^2)/x_o$, which is consistent with the prior results for ψ and χ Eqs. (31),(32). This is not surprising, since the specification of a quantum state (such as the Hartle-Hawking state) should uniquely determine these Green's functions [43].

With the new ansatz being rigorously stipulated, we are now in a position to re-evaluate the extremal black hole properties. These results are reported below with commentary wherever clarity is required:

$$m(x)=m_0+2\frac{\mathcal{K}}{l^2}\left\{2x-6\frac{x_o^2}{x}+2\frac{x_o^3}{x^2}-16\,\frac{x_o^2}{x+x_o}+\left(3x+6\frac{x_o^2}{x}-\frac{x_o^4}{x^3}\right)\left[\ln\left(\frac{(x+x_o)^2}{lx}\right)+\Theta\right]\right\}, \tag{75}$$

where we have redefined

$$\Theta\equiv\ln\left(\frac{L-x_o}{L+x_o}\right)-2\frac{x_oL}{L^2-x_o^2}+\frac{1}{2}\ln\mu^2 \tag{76}$$

and m_0 is a constant that must be constrained to satisfy $m(x_o)=0$. This constraint becomes evident when we consider a first-order Taylor expansion of $g(x_o+\Delta x_o)$. Note that no such method of fixing m_0 is apparent in the non-extremal analysis:

$$\varpi(x)=-\frac{1}{x}+\frac{4}{x+x_o}-8\frac{x_o}{(x+x_o)^2}+\frac{8}{3}\,\frac{x_o^2}{(x+x_o)^3}+\frac{6}{x}\left[\ln\left(\frac{(x+x_o)^2}{lx}\right)+\Theta\right], \tag{77}$$

$$R(x)=-2\frac{x^4+3x_o^4}{l^2x^4}+4\frac{\mathcal{K}}{lx}\left\{12\frac{x+x_o}{x}+6\frac{x_o}{x^3}(3x^2+x_o^2)+8\frac{x_o^2}{x^4}(x^2+x_o^2)+8\frac{x_o}{x}\,\frac{(3x+x_o)(x^2+x_o^2)}{(x+x_o)^3}+\frac{3}{x^4}(3x^2+x_o^2)(x^2-x_o^2)\times\left[\ln\left(\frac{(x+x_o)^2}{lx}\right)+\Theta\right]\right\}, \tag{78}$$

$$\Delta x_o=l\,\frac{m'}{g''_{CL}}\bigg|_{x=x_o}=2\mathcal{K}l. \tag{79}$$

For this calculation, we have considered a first-order Taylor expansion of $g'(x_o+\Delta x_o)$, since such an expansion for g leaves Δx_o as an indeterminate quantity. If we impose the

[8] It has been argued [17] that the same conditions apply to the extremal case, in spite of the difficulties in formalizing an extremal analogue to ''Kruskal-like'' (i.e., free-falling) coordinates.

one-loop constraint $g'(x_o+\Delta x_o)=0$ (which is justified by Hawking's conjecture [5]: an extremal black hole should retain its nature, regardless of radiation effects) on the expansion in question, then Eq. (79) follows:

$$T=\beta^{-1}\rightarrow\text{indeterminate (arbitrary by hypothesis)}, \tag{80}$$

$$E_{sub}=2\frac{x_o^2}{l^2L}+\frac{lLm(L)}{L^2-x_o^2}+2\frac{\mathcal{K}x_o}{lL^2(L^2-x_o^2)}\times[5L^3-7x_oL^2-5x_o^2L-x_o^3], \tag{81}$$

$$S=0. \tag{82}$$

This result of vanishing entropy occurs trivially, as the on-shell Euclidean action is now linearly proportional to β. Furthermore, the horizon surface term (which normally accounts for the entropy) must vanish according to $g(x_q)=g'(x_q)=0$ [cf. Eq. (45)]:

$$\gamma_J=\frac{l^2J}{2Lx_o^2}\left[1+\frac{1}{2}\frac{l^3L^2}{L^2-x_o^2}\left(\frac{m(L)}{L^2-x_o^2}-\frac{8\mathcal{K}}{l^2x_o}\right)\right], \tag{83}$$

$$T_{uu}=-\frac{\mathcal{K}}{l^4}\frac{(x^2-x_o^2)^4}{x^6}\left\{2\frac{x_o}{(x+x_o)^4}(3x^3-2x_ox^2-5x_o^2x-2x_o^3)-3\left[\ln\left(\frac{(x+x_o)^2}{lx}\right)+\Theta-\frac{1}{2}\right]\right\}, \tag{84}$$

$$T_{uv}=8\frac{\mathcal{K}}{l^4}\frac{(x^2-x_o^2)^2}{x^2}. \tag{85}$$

Evidently, the approach of this section is a substantial improvement over the prior limiting procedure. All properties (except arbitrary temperature) are now well defined and all local quantities are regular throughout the relevant manifold. Furthermore, the stress tensor satisfies the horizon regularity conditions (60), which implies that our choice of boundary conditions (74) appropriately describes an extremal Hartle-Hawking state.

VI. CONCLUSION

In the preceding sections, we have examined numerous properties of a spinning BTZ black hole in a state of thermal equilibrium. An analytical description of the one-loop back reaction was formulated with the application of perturbative techniques to a dimensionally reduced model. The one-loop thermodynamics was extracted from the on-shell Euclidean action, which effectively describes the partition function in a semi-classical regime. When considerations were limited to non-extremal black holes, we found these geometrical and thermodynamic calculations to be both regular and unambiguously defined. However, the extremal limit of these calculations was shown to be plagued by divergent behavior. This extremal breakdown in the one-loop approximation is suggestive of a generalized third law of thermodynamics that prohibits continuous evolution from non-extremal to extremal states.

As an alternative to the limiting procedure, we have also considered the extremal case from the following viewpoint: extremal and non-extremal black holes are qualitatively distinct entities. In this alternative approach, the extremal solution was assumed from the beginning and horizon regularity (in the one-loop geometry) was used to fix the boundary conditions. With this procedure, we found all calculations to be regular and all thermodynamic properties (with one exception) to be well defined. The one exception was temperature, which we justifiably regarded as an arbitrary quantity. Other notable results were a vanishing entropy and the horizon regularity of the stress tensor in the free-falling frame. Although this analysis was limited to the study of BTZ black holes, qualitatively similar outcomes have been obtained for the Reissner-Nordström case [13].

The arbitrary nature of the extremal temperature is somewhat unsettling inasmuch as the physical state is (at least in some sense) a thermal one with non-vanishing asymptotic radiation. To help clarify this apparent conflict, we take note of recent findings by Liberati *et al.* [14]. They have considered an extremal Reissner-Nordström black hole undergoing collapse and demonstrated that (in spite of asymptotic particle production) the temperature remains undefinable on account of a non-Planckian distribution. Although this result does not apply directly to static BTZ black holes, it does imply an intrinsic elusiveness in measuring the temperature of an extremal black hole.

One may find it intuitively disturbing to assign a vanishing entropy to a macroscopic object that emits radiation, although a strong case for this has been recently put forth. Hod [15] argued in favor of $S_{ext}=0$ by appealing to the second law of thermodynamics on the basis of a *Gedanken* experiment. However, before any definitive viewpoint can be reached on this subject, we will ultimately require a clearer picture of what degrees of freedom underlie black hole entropy.

A couple of final technical notes regarding our results are in order. First, by imposing an axially symmetric reduction on the $(2+1)$-dimensional action, we have studied a truncated form of the one-loop effective action for which only the "*s* waves" of the matter fields are quantized. It is further significant that, because of an anomalous reduction process [27], such a truncated form may not accurately describe even the *s* waves. Hence, from a 3D point of view, the quantum effective action should only be regarded as an approximation. That is, modifications may still be required if our results and conclusions are to be directly applied to the higher-dimensional theory. However, from the viewpoint of $(1+1)$-dilaton black holes, this dimensional-reduction anomaly can be considered as inconsequential.

Second, there has been an omission of certain non-local terms in the conformally invariant portion of the effective action [as discussed after Eq. (4); also see Ref. [36]]. The inclusion of these terms would likely modify the quantitative details of our one-loop calculations. We expect, however, that the qualitative outcomes of this paper will persist even after the conformally invariant terms have been rigorously dealt with. That is, we anticipate that extremal black holes will maintain their regular behavior when the methodology of Sec. V is applied, whereas the singularities arising in the extremal limit (of the non-extremal calculations) will persevere. To partially justify this last statement, we note that an expansion based on ''covariant perturbation theory'' (the process by which the relevant terms have been derived) is only expected to be valid when the derivative of the curvature is much larger than the curvature itself [44]. This appears to be an appropriate condition with respect to (for instance) the reduced Reissner-Nordström black hole; however, this is not the case for the reduced BTZ black hole, which is essentially a theory of constant-curvature gravity.

It is our hope that the above issues will be formally addressed in a future work. In any event, the techniques of our current analysis should prove useful in subsequent studies on both extremal and non-extremal black hole thermodynamics.

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